Low Pass Filtering of Gravity Field Models
by Gently Cutting
the Spherical Harmonic Coefficients of Higher Degrees

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If a gravity field model represented by spherical harmonics up to the maximum degree $n = N_{max}$ is analysed by not using all coefficients C_{nm} and S_{nm} but cutting the model at $n = N < N_{max}$ (or setting to zero all coefficients for $n > N$), then this corresponds to a low pass filtering in the frequency domain. Unfortunately, this 'rigorous' cutting leads to the well-known side lobes in the spatial structures of the truncated fields. The mathematical explanation for this is the following: The low pass filtering by 'rigorous' cutting of the short wavelengths corresponds to a multiplication of the model with a boxcar function in the frequency domain. However the Fourier transform of the boxcar function is the slit function (and vice versa). Both functions form a Fourier transform pair (see Fig. 1).

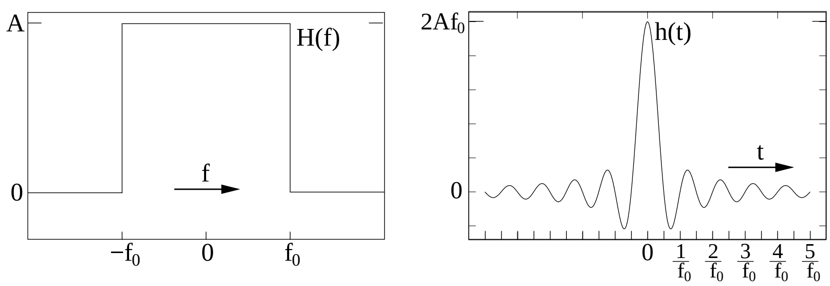

Figure 1: The Fourier transform pair:
boxcar function in the frequency domain (left) and
slit function in the time domain (right)

Thus the Fourier (back)transform of the function $H(f)$ in the frequency domain

$$\begin{aligned} H(f) &= A \quad \text{for} \quad |f| < f_0 \\ H(f) &= \frac{1}{2} A \quad \text{for} \quad |f| = f_0 \\ H(f) &= 0 \quad \text{for} \quad |f| > f_0 \end{aligned} \tag{1}$$

is the function $h(t)$ in the spatial or time domain

$$h(t) = 2Af_0 \ \frac{sin(2\pi f_0 t)}{2\pi f_0 t} \tag{2}$$

As an example for these side lobe effects, figure 2 shows the gravity anomalies of the model 'EIGEN_GRACE01S' truncated at $N = 90$. One can see clearly the 'ring waves' around the biggest amplitudes of the gravity anomalies (Andes, Hawaii, deep-sea trenches).

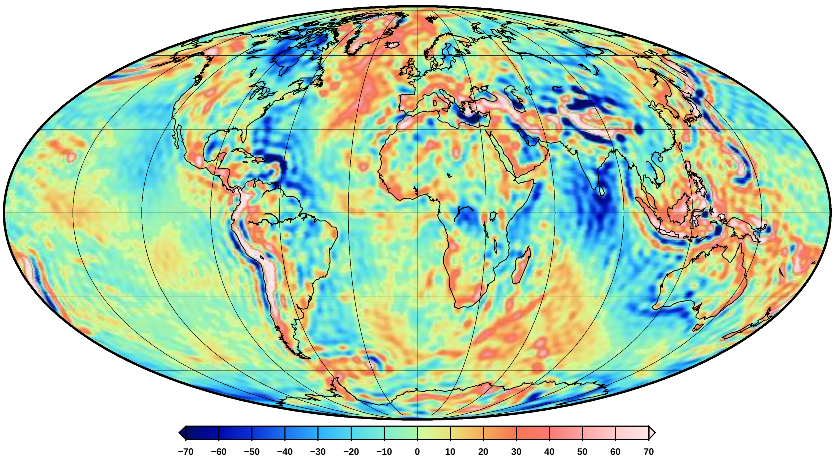

Figure 2: Gravity anomalies of the model EIGEN_GRACE01S
truncated at N=90

A 'gentle' truncation in the frequency domain can minimize this effect. To accomplish this, a function $f(x)$ has to be looked for, which decreases monotonicallyly from 1 to 0 in the interval x_a to x_b and has horizontal first derivatives at the points x_a and x_b:

$$f(x_a) = 1 \ ; \quad f(x_b) = 0 \ ; \quad f'(x_a) = f'(x_b) = 0 \tag{3}$$

Using the simple ansatz

$$\begin{aligned} f(x) &= C_4 x^4 + C_2 x^2 + C_1 x + C_0 \\ f'(x) &= 4C_4 x^3 + 2C_2 x + C_1 \end{aligned} \tag{4}$$

it follows:

$$\begin{aligned} C_4 x_a^4 + C_2 x_a^2 + C_1 x_a + C_0 &= 1 \\ C_4 x_b^4 + C_2 x_b^2 + C_1 x_b + C_0 &= 0 \\ 4C_4 x_a^3 + 2C_2 x_a + C_1 &= 0 \\ 4C_4 x_b^3 + 2C_2 x_b + C_1 &= 0 \end{aligned} \tag{5}$$

and:

$$f(x) = \left(\frac{x - x_a}{x_b - x_a}\right)^4 - 2\left(\frac{x - x_a}{x_b - x_a}\right)^2 + 1 \tag{6}$$

can be found. Figure 3 shows the function from $x_a = 60$ to $x_b = 120$.

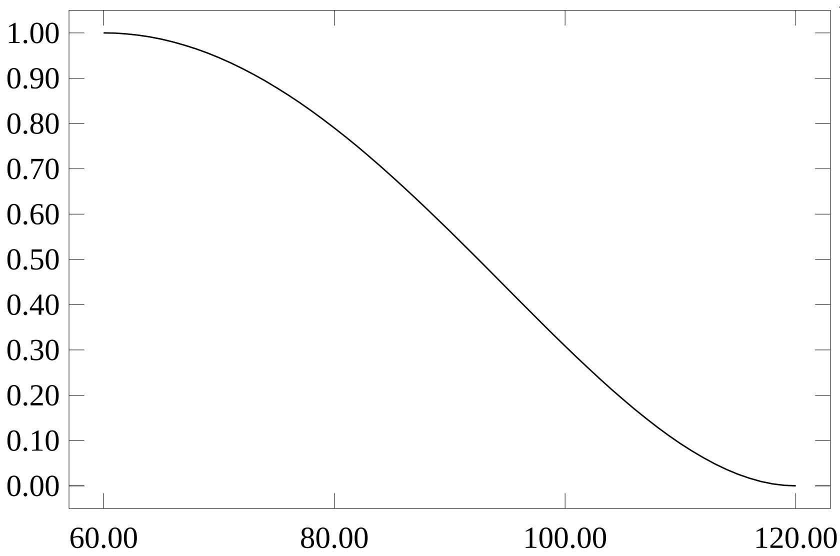

Figure 3: Gently cut function for $x_a = 60$ and $x_b = 120$

Figure 4 shows the result after replacing the 'rigorous' truncation of the gravity field model 'EIGEN_GRACE01S' at $N = 90$ by 'gently' cutting the spherical harmonic series from $N_a = 60$ to $N_b = 120$ (see Fig. 3), i.e. the coefficients C_{nm} and S_{nm} for $60 < n < 120$ have been multiplied by the function f(n) (eq. 6). The unfiltered model 'EIGEN_GRACE01S' ($N_{max} = 140$) is shown in figure 5.

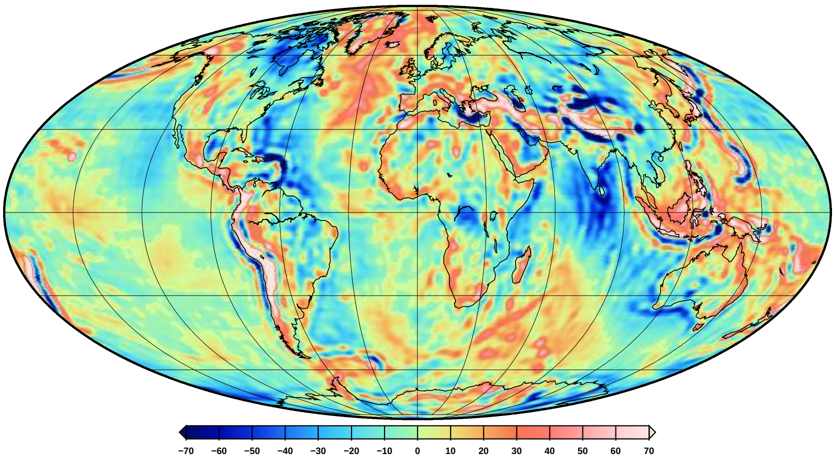

Figure 4: Gravity anomalies of the model EIGEN_GRACE01S gently cut from $N = 60$ to $N = 120$

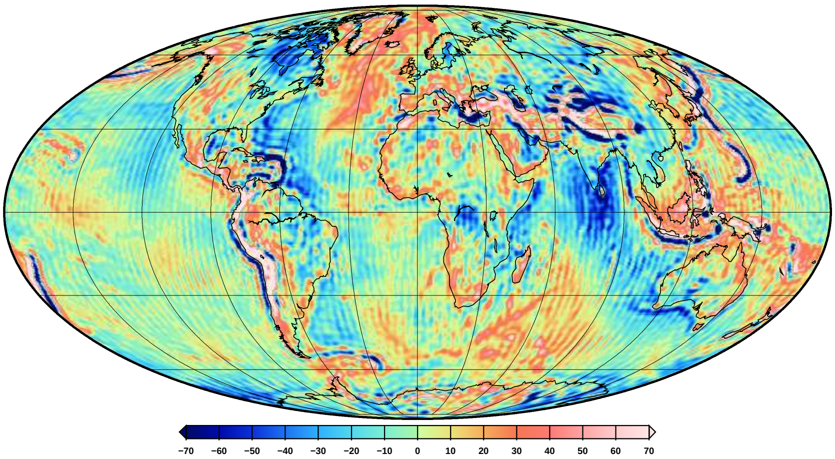

Figure 5: Gravity anomalies of the model EIGEN_GRACE01S up to the full degree $N = 140$